The Beauty of Mathematics – A Rough Sketch for a Proof

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No non-poetic account of reality can be complete. [i]

John Myhill

Mathematics and the aesthetics enjoy a long history of mutual references, which may be explained by the fact that both domains are dealing with relations: structure, content, schemata and (dis)similarity (cf. Wechsler 1978). And beauty? Beauty is the experience of pleasure and amazement from something that is aesthetic. Although "theorems and proofs which are agreed upon to be beautiful are rare" (Rota 1997), many mathematicians describe the process and/or the results of their work as beautiful, giving them and others an aesthetic experience. The French universalist Jules Henri. Poincare (1854-1912) suggested[ii] that: "the mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful." In addition to numerous examples of beautiful images, pattern and visualisations in mathematics, or generated with the help of mathematics, there is the question whether aesthetic elements are a vital component in the process that generates mathematical results. Poincaré was one of the first mathematicians to draw attention to the aesthetic dimension of mathematical invention and creation. For him the aesthetic plays a major role in the subconscious operations in a mathematician's mind (see Hadamard 1945).

In this essay I am going beyond a general discussion of how beautiful mathematics is and instead I will try to pin down an aesthetic element in a proof, providing support for the view that that the distinguishing feature of the mathematical mind is not logical but aesthetic. My investigation is motivated by Nelson Goodman's theory of symbols (1976). Although I was not able to map all of Goodman's symptoms of the aesthetic (syntactic density, semantic density, syntactic repleteness and exemplification) onto the notion of proofs in a satisfactory way, I believe that the distinction between *denotation* and *exemplification* in the proof for the irrationality of $\sqrt{2}$ allows a more focussed discussion about the *evaluative* and *generative* role of the aesthetics in mathematics. Furthermore, I also suggest that the notion of exemplification in the sense of Goodman reveals *ambiguity*, which subsequently plays a positive role in the development of the proof (cf. Byers 2007).

How beautiful!

The English mathematician Godfrey Harold Hardy (1877-1947) wrote in 1940: "The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics." It is the concise form and the model character that suggests comparisons of

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equations and proofs with poems[iii]. Great equations, as great poems, serve as an inspiration and are considered a revelation. Albert Einstein (1879-1955) allegedly said "Pure mathematics is, in its way, the poetry of logical ideas." and the German mathematician Karl Weierstraß (1815-1897) is said to have declared, "A mathematician who is not at the same time something of a poet will never be a full mathematician." Consider Einstein's 1905 equation $E=m\cdot c^2$, which reveals that matter is energy. Its beauty is related to the fact that only six symbols suffice to describe something universal. Similar to a great poem, very little could be changed without spoiling it (Farmelo 2002). In fact, in this case one could only remove the multiplication dot. The English theoretical physicist Paul Dirac (1902-1984), when asked to summarise his philosophy of physics, said "Physical laws should have mathematical beauty" (Farmelo 2002).

Apart from poetry, mathematics is frequently linked to music. The French mathematician Jean Dieudonné gave one of his books the title: *Mathematics – The music of reason.* Timothy Gowers (born 1963), recipient of the Fields Medal for Mathematics writes (2002): "It often puzzles people when mathematicians use words like 'elegant', 'beautiful', or even 'witty' to describe proofs, but an example such as this gives an idea of what they mean. Music provides a useful analogy: we may be entranced when a piece of moves in an unexpected harmonic direction that later comes to seem wonderfully appropriate, or when an orchestral texture appears to be more than the sum of its parts in a way that we do not fully understand. Mathematical proofs can provide a similar pleasure with sudden revelations, expected yet natural ideas, and intriguing hints that there is more to be discovered. Of course, beauty in mathematics is not the same as beauty in music, but then neither is musical beauty the same as the beauty of a painting, or a poem, or a human face."

The mathematician Gian-Carlo Rota (1932-1999) went further than others and argued that "The beauty of a theorem is a property of the theorem, on a par with its truth or falsehood. Mathematical beauty and mathematical truth are not to be distinguished by labeling the first as a subjective phenomenon and the second as an objective phenomenon. Both the truth of a theorem and its beauty are equally objective qualities, equally observable characteristics of a piece of mathematics which are equally shared and agreed upon by the community of mathematicians. The truth of a theorem does not differ from its beauty by a greater degree of objectivity; rather, the distinction between truth and beauty in mathematics is made on the basis of their properties of truth and beauty, when viewed as worldly phenomena in an objective world." (Rota 1997). His discussion of these issues is highly recommended.

The theme of mathematics and the aesthetics has been discussed in various directions and I shall not attempt to be comprehensive here. In addition to the references given in the text, I would therefore like to point at the recent edited volume *Mathematics and the Aesthetic* (Sinclair *et al.* 2006), in which ten authors explore different ways in which mathematics is an aesthetic experience. In what follows I am going to focus on a particular mathematical proof in which two mathematically equivalent equations have two distinct roles of reference, each of which is crucial to the proof in different ways. The semantics of these two equations can be explained in terms of Nelson Goodman's theory of symbols. This result will then form the basis for a more general discussion of the aesthetics in mathematics.

A Beautiful Proof

In his famous essay *A Mathematican's Apology*, the English mathematician G.H. Hardy (1940) considered the proof for the *irrationality* of $\sqrt{2}$ as an example of mathematical beauty. He attributes this proof to Pythagoras and points out that, in a more general form, a proof can also be found in Euclid's *Elements*. There exist several versions of the proof, including a graphical/geometric one (Apostol 2000). I use the form in which it is most frequently presented (see Gowers 2002).

The theorem says "$\sqrt{2}$ is irrational" and it can be (dis)proven using a technique known as *reductio ad absurdum*[iv]:

1. A number is called *rational* if it can be written as a fraction p/q, where p and q are whole numbers, and *irrational* if it cannot.
2. If $\sqrt{2}$ is (were) rational, we can find two whole numbers p and q such that
   $\sqrt{2} = p/q$
   where p/q is an irreducible fraction with p and q having no common factor.
3. Any fraction p/q is equal to some fraction a/b, where a and b are not both even.
4. Therefore, if $\sqrt{2}$ is (were) rational, then we can find whole numbers a and b, not both even, such that
   $\sqrt{2} = a/b$ (i)
5. Multiplying both sides of (i) with b, gives
   $\sqrt{2} \cdot b = a$
   Squaring both sides
   $2 \cdot b^2 = a^2$
   or
   $a^2 = 2 \cdot b^2$ (ii)
   Remark: To say that "$\sqrt{2}$ is irrational" is then obviously the same as saying "2 cannot be expressed in the form $(a/b)^2$".
6. From (ii) it follows that a^2 must be even.
   Remark: a^2 is even because it is equal to $2 \cdot b^2$, which is divisible by 2. Numbers divisible by 2 are even by definition[v].
7. Because a^2 is even, a is even.
   Remark: This is the contrapositive[vi] of the fact that the square of an odd number is odd. One could formulate this in form of a theorem: Suppose that a is an odd integer, then a^2 is an odd integer. *Proof:* Assume the contrary by supposing that a is an odd integer but that the conclusion is false, i.e., a^2 is an even integer. As a is odd, $a = 2c + 1$ for some integer c.
   Thus $a^2 = (2c+1)^2 = 4c + 2c + 1$, which contradicts that a^2 is even. Thus our assumption that a^2 is even must be wrong, i.e., a^2 must be odd.
8. If a is even then $a = 2 \cdot c$ for some integer value c.
9. Substituting $2 \cdot c$ for a in (ii) gives $(2c)^2 = 2 \cdot b^2$ and equivalently $b^2 = 2 \cdot c^2$.
10. From $b^2 = 2 \cdot c^2$ it follows that b^2 and hence also b is even.

Final step: We have shown that a and b must both be even. This means they have the common factor 2. This however contradicts our initial hypothesis ("$\sqrt{2}$ is rational and can be written in the irreducible form a/b, where a and b cannot be both even") and we conclude that $\sqrt{2}$ is an *irrational number*[vii]. *Q.E.D.*[viii]

A proof shines a light on an answer, gives certainty and has the potential to increase understanding of other problems. The result is a revelation; it answers the question once and for all – and if done elegantly, beyond doubt. Even reading the proof above gives a sense of pleasure, let alone conducting one. The French mathematician André Weil (1906-1998) described the experience as "the state of lucid exaltation in which one thought succeeds another as if miraculously, and in which the unconscious seems to play a role [..] unlike sexual pleasure, this feeling may last for hours at a time, even for days. Once you have experienced it, you are eager to repeat [..]"[ix]

Mathematical Invention

The rearrangement of Equation (i) into (ii) really is simple, and can be conducted with no creativity through the mechanical (algorithmic) manipulation of symbols. However, to recognise that (ii) symbolizes that a is an even number, requires that one was told this before (and remembers it) or, one "sees" it while doing the proof. My suspicion is that those mathematicians everyone admires are not only good in applying rules in an appropriate order but they also "see" things that are hidden to most of us. There is often a particular step that is considered beautiful[x] and although the proof discussed here is not a good example, the realisation that (ii) defines an even number, is the crucial step towards the insight that the proof as a whole gives. While afterwards things may seem obvious (logical), in the process of deriving a proof for the first time yourself, the realisation may harbour a pleasant experience. Because most people only read proofs, rather than generating them (and thus missing out on the erotic side-effects), proofs are often thought of as a *demonstration*, rather than an *experiment*. While a demonstration has the air of a dull run-down of instructions (that demonstrate the truth or validity of something narrowly defined), an experiment is something more exciting because it is engaging, creative and is usually set in a wider scientific context related to a question of general interest.

What counts for the "success" of a proof is not only the final result, a proven theorem, but also the arrangement and presentation of the proof – to make it transparent, evident and compelling. Going through the proof above and various alternative presentations of it, one notices that it is largely a composition of ideas and intermediate results/parts, linked together, that make up the proof as a whole. Sinclair (2004) distinguishes between the *evaluative* and the *generative* roles of the aesthetic: the former is involved in judgements about beauty, elegance, and significance of entities such as proofs and theorems, while the generative role of the aesthetic is a guiding one, responsible for the generation of new ideas and insights. I am going to argue further below that the perception of the proof as a whole (not just the fact that a theorem is proven) is related to the evaluative role of the aesthetics, while the distinction between denotation and exemplification in the sense of Goodman is related to the generative role of the aesthetic in composing the proof.

"Hadamard, following Poincaré, distinguishes in mathematical work a conscious stage of *preparation*, an unconscious stage of elaboration or *incubation*, and *illumination* that reverts to conscious thinking, and a conscious stage of *verification*. The incubation stage is described as combinatorial in nature: ideas are put together in various ways until the right combination is chosen. And it is claimed that this choice is made on an *aesthetic* basis." (Ruelle 2007, page 86) "According to Poincaré, two operations take place in mathematical invention: the construction of possible combinations of ideas and the selection of the fruitful ones. Thus, to invent is to choose useful combinations from the numerous ones available; these are precisely the most beautiful, those best able to "charm this special sensibility that all mathematicians know'." (Sinclair 2004). In the words of Hadamard (1945): "invention is choice" and "this choice is imperatively governed by the sense of scientific beauty." The process of choosing alternatives implies the possibility of mistakes. However, even a "false" idea can be valuable. Byers (2007, page 256) tells the story of Yutaka Taniyama speaking about Goro Shimura (both made important contributions to the proof of Fermat's Last Theorem): "He was gifted with the special capability of making many mistakes, mostly in the right direction. I envied him for this and tried in vain to imitate him, but found it quite difficult to make good mistakes." Some of the greatest discoveries are glorious "failures" and this applies to any aspect of life and the sciences. The development of drugs in medicine provides many examples of chance discoveries, Viagra being a well-known example.

Aesthetic Elements in the Proof

We can now proceed to discuss aesthetic elements in the proof for the irrationality of the square root of two. A related discussion on the beauty of this proof, not however linked to Goodman's theory of symbols, can be found in Papert (1978), Dreyfus and Eisenberg (1986) as well as King (1992). The philosopher Gerhard Heinzmann[xi] (Nancy-Université / CNRS, France) pointed out that the two key elements of the proof

(i) $\sqrt{2} = a / b$

and

(ii) $a^2 = 2 \cdot b^2$

can be further subjected to an analysis in the sense of Nelson Goodman's *Languages of Art* (1976). The notion of *reference* can be expressed in two different modes: *denotation* and *exemplification*. According to Gerhard Heinzmann, equation (i) serves as an instance of the predicate "=" and is thus denotation, while (ii) is exemplification of the predicate "even number". He described exemplification as the first aesthetic element in the proof; the second is the elimination of redundancy. This is related to Goodman's concepts of "syntactic and semantic density" as well as "syntactic repleteness". Exemplification and simplicity are not independent: The exemplifying notation of (ii) was suggested by the improved degree of saturation of the proof.[xii] At this point we note the two different interpretations of (i) and (ii) and emphasize that although mathematically equivalent, both perspectives were crucial in the proof. Although essential to his theory of symbols, reading Goodman (1976) I could not find clear and concise definitions of denotation and exemplification, which could easily be mapped onto the mathematical setting. I therefore base my arguments on the thorough recent discussion by Vermeulen *et al.* (2009).

Nelson Goodman uses 'reference' in the general sense of "standing for" and then distinguishes and compares different kinds of references. Denotation is reference from a symbol (here the two equations) to one or many objects (here the predicates "rational number" and "even number") it applies to. Denoting symbols are called *labels*, while symbols that exemplify are referred to as *samples*. Exemplification runs in the opposite direction and is reference from an object back to a label that applies to it. Let us consider equations (i) and (ii) as symbols, potentially samples, which refer to the predicates (labels) P1: "rational number" and P2: "even number" respectively. We have two features of x exemplifying y:

(iii) C1: y denotes x and C2: x refers to y

Condition (C1) is necessary but not sufficient for exemplification. Condition (C2) requires that an object refers to the label it exemplifies. Only if y denotes x, may x exemplify y. In our context, both conditions appear to be satisfied: (P1) denotes (i) and (i) refers to (P1); (P2) denotes (ii) and (ii) refers to (P2). Rather than giving a definition for exemplification, Vermeulen *et al.* argue that in addition to 'reference' and 'denotation', 'exemplification' is the third basic notion of Goodman's theory of symbols. Exemplification is then introduced by *explication*[xiii]. This leads to a revision of (iii) in the following form:

(iv) x exemplifies $y \leftrightarrow y$ denotes $x \wedge x$ exemplifies y.

(v) x exemplifies $y \rightarrow y$ denotes $x \wedge x$ refers to y.

Condition (iv) emphasizes that denotation in the opposite direction is only a necessary condition for exemplification, while (v) provides an alternative, conditional instead of biconditional, reading of (iii). Applying this to our proof, we find that from (v) that (ii), $a^2 = 2 \cdot b^2$, exemplifies an "even number", (P2), because (P2) denotes (ii) and (ii) also refers to (P2). However, although one might argue that (i) refers to or even denotes a rational number, (P1); from (iv), it follows that (P1), "rational number", does not exemplify (i) since $\sqrt{2}$ is not rational (as shown). In Chapter II of his *Languages of Art,* Goodman introduces exemplification in another related way via *possession*. It states two conditions for *x* exemplifying the property *y*:

(vi) C3: *x* possesses *y* and C2: *x* refers to *y*.

With this reading, (ii) possesses the property (P2) but (i) does not possess the property of being rational number (P1).

Both equations, (i) and (ii), can be looked at as algebraic/arithmetic propositions when considered equations but (ii) in addition makes an exemplifying assertion that is necessary for the proof "to succeed". The notion of exemplification therefore allows a source of meaning in addition to denotation. "As for the features that an artwork appears to exemplify despite its not, literally, possessing them (as when, for instance, a painting is claimed to express sadness in spite of the fact that paintings cannot literally be sad). Goodman claims that such features are *metaphorically exemplified*, or *expressed*. In brief, a work of art expresses something when it metaphorically exemplifies it. Like representation, exemplification and expression are relative, in particular they are relative to established use (Goodman 1976, page 48).

As William Byers (2007) points out, the notion of equality, symbolized by "=", is surprisingly multifaceted. Doesn't "=" simply mean that both sides are equal/identical? The symbol itself does not tell us whether reading it aloud we should read it with a "!"or a "?" behind it, whether the expression holds true or not. Equations (i) and (ii) are in one sense identical and thus exchangeable – one can stand for the other, in the sense that we got (ii) through a simple manipulation of (i). Equation (i) is a hypothesis (false as it turns out), while (ii) is a definition (of an even number). Within a proof, an equation can be a starting point (definition, assumption, as (i) in our proof above) but also an end-point, a result (e.g. $E = m \cdot c^2$). "One could say that an equation is a mathematical idea that carries with it one or more optimal contexts that give it meaning." (Byers 2007, page 211). In the same way, the equation $E = m \cdot c^2$ may simply be seen as an ordinary arithmetical proposition – plug in the value for m and voilà - out comes the value for E, or, the same equation is considered a metamathematical assertion (formulating a natural law of physics, explaining matter as energy).

What our discussion shows is that there is *ambiguity*[xiv] in the interpretation of (i) and (ii) in the context of the proof[xv]. While the formal system of the equations and their manipulation are unambiguous, ambiguity results from the two equivalent equations (i) and (ii) having different meanings and roles in the proof. This ambiguity between the meanings not only makes the proof possible, more generally it enriches our aesthetic experience of mathematical reasoning: The difference between denotation in (i) and exemplification in (ii) is something that cannot be resolved by the syntax of the expressions alone. If we understand this difference as a source of ambiguity, we could then follow William Byers (2007), who argues that ambiguity is a crucial mechanism in mathematics[xvi]: "This ambiguity is neither accidental nor deliberate but an essential characteristic of the conceptual development of the subject as well as of the person attempting to master the subject. The ambiguity is not resolved by designating one meaning or one point of view as correct and then suppressing the others. The ambiguity is "resolved" by the creation of a larger meaning that contains the original meaning and reduces to them in special cases. This process requires a creative act of understanding or insight. Thus ambiguity can be the doorway to understanding, the doorway to creativity." (page 77). Byers quotes the composer Bernstein speaking on music: "The more ambiguous, the more expressive." Instead of pursuing this line further we return to the philosophy of Nelson Goodman who inspired the analysis above.

Goodman's Aesthetics

Nelson Goodman (1906-1998) advocated a form of *cognitivism*: by using symbols we discover (indeed we build) the worlds we live in, and the interest we have in symbols – artworks amongst them – is distinctively cognitive (Goodman 1976). To Goodman, aesthetics is but a branch of epistemology (cf. Giovannelli 2005). Goodman's epistemological position may thus be linked to *conventionalism*, the philosophical attitude that fundamental principles of a certain kind are grounded on (explicit or implicit) agreements in society, rather than on external reality[xvii].

The aim of Goodman's aesthetics is to discover or analyse *structures of appearance*. This appearance of things is the basis for our everyday reasoning. According to Goodman we *project* predicates onto the world we live in and thereby *construct* reality. Goodman says that there is no difference in principle between the predicates we use and those we could use, but rather a pragmatic difference in habit, or of "entrenchment" of certain predicates and not others. Understanding the worlds of art is then not very different, from understanding the worlds of science or of ordinary perception: it requires *interpretation* of the various symbols involved in those areas and identifying those symbols which are successfully projected.

In addition to his work on the language of art, and based on a paper Goodman wrote together with the logician Quine, he is also associated with founding a contemporary version of *nominalism*, which argues that philosophy, logic, and mathematics should dispense with set theory. According to Thomas Tymoczko (1998), Quine had "urged that we abandon *ad hoc* devices distinguishing mathematics from science and just accept the resulting assimilation", putting the "key burden on the theories (networks of sentences) that we accept, not on the individual sentences whose significance can change dramatically depending on their theoretical context." In doing so, Tymoczko claimed, philosophy of mathematics and philosophy of science were merged into quasi-empiricism: the emphasis of mathematical practice as effectively part of the scientific method, an emphasis on method over result (Wikipedia 2008).

In a 1968 critique titled *The Activity of Aesthetic Experience*, Goodman criticises the "domineering dichotomy between the cognitive and the emotive". Rather than looking for a sharp criterion of the aesthetic ("What is art?"), Goodman suggests to examine the aesthetic relevance of symbol processes ("When is art?"): There is art whenever (only and on every occasion when) and object exhibits the following symptoms of the aesthetic: syntactic density, semantic density, syntactic repleteness, and a symptom that "distinguishes exemplificational from denotational systems."

Stated briefly, syntactic density describes continuous processes in which smallest changes are significant, while "semantic density" describes the continuous character of the signified. Relative syntactic repleteness refers to the case where relatively many syntactical features are semantically relevant. The usual example given to illustrate Goodman's notion of syntactic density is the distinction between pictures and diagrams. A "pictorial symbol system" is said to be *syntactically and semantically dense*. That is, given any two marks in a picture, no matter how small the difference between them, they could be instantiating two different characters (symbols), and given any two characters, no matter how small the difference between them, they may have different referents (Goodman 1976, page 226-227). In pictures any difference may make a difference. Even a simple picture is dense, in the sense that any, however small, mark on the canvas may turn out being relevant to pictorial meaning.

In a technical data plot or a cartographic map the relation of objects to each other are more important then say the style in which lines are connected. In a picture, produced as a piece of art, this difference can matter. In (Gennette and Goshgarian 1999, page 34) a nice story is reported according to which during World War I, Italian customs officers detained Stravinsky because he was carrying a cubist portrait of himself done by Picasso: they refused to let him cross the boarder because they were convinced the portrait was a map, possibly of strategic importance.

Syntax, Semantics, and Pragmatics

The proof and its discussion highlights the fact that mathematical reasoning is not just an algorithmic manipulation of symbols, according to syntactic rules but that in mathematical reasoning the semantics of equations play a crucial role in generating understanding. Let us therefore look into the relation between syntax and semantics in mathematical (not logical) reasoning[xviii]. *Semantics* is about the relation between signs and the things they refer to, their denotata, while *syntactics* is about the elation of signs to each other in formal structures. As is often the case in science, these two definitions suggest a framework devoid of someone making use (or sense) of it. It is thus prudent to consider a third aspect, which we may refer to as *pragmatics* – studying the relation of signs to their impacts on those who use them. Pragmatics has to do with the ways in which context contributes to meaning.

In logic, formal symbols are mere marks and their interpretation is considered irrelevant to the process of reasoning. In the proof for the irrationality of $\sqrt{2}$, this is how we arrived at (ii) from "mindless" manipulation of (i). Then however we interpreted (ii) differently to (i) and were able to proceed with the proof. Both interpretations were required for the mathematical proof "to make sense". The difference between (i) and (ii) could be explained in terms of the *reference relation* within them. This difference between denotation and exemplification in (i) and (ii) is, however, not determined by the symbol "=" and neither by the syntax of the equations.

Most people associate with mathematical reasoning a logical, rational manipulation of equations that leaves little or no space for interpretation. But are semantics really irrelevant to proofs? If not, in what way? The American mathematician Edward Nelson (2002) provides one position: "Mathematics is expressed in terms of formulas, which are strings of symbols of various kinds put together according to certain rules. As to whether a string of symbols is a formula or not, there is no dispute: one simply checks the rules of formation. Certain formulas are chosen as axioms. Here there is great scope for imagination and inspiration from one semantics or another, to choose fruitful axioms. Certain rules of inference are specified, allowing one to deduce a formula as conclusion from one or two formulas as premise or premises. Then a proof is a string of formulas such that each one is either an axiom or follows from one or two preceding formulas by a rule of inference. As to whether or not a string of formulas is a proof there is no dispute: one simply checks the rules of formation. This is the syntax of mathematics. Is that all there is to mathematics? Yes, and it is enough. [..] Now in defense of semantics it can be said that it is a useful source of inspiration and that it is essential in pedagogy." Another, (my) perspective is to agree with Nelson only with respect to the check of a proof but not its creation.

The Heinzmann-motivated argument for aesthetic elements in proofs does also appear to be a stronger (more specific) "proof" than the suggestion that a proof is beautiful because of its "elegance" or the wide use that follows from it. A particular version of a proof[xix] is often considered "elegant" if it is concise, that is, it uses a minimum of axioms, assumptions or previous results and other theorems. To use Goodman's terminology, a proof is *syntactically dense* if no or very few elements of it could be changed, without changing the function of the whole representational system. Syntactic density can thus serve as a criterion for simplicity and the reduction of redundant elements.

A proof can also be considered elegant if the method of proof can be generalised, the proof is obtained in a surprising (non-obvious) way or the result itself is considered a surprise. In his *A Mathematician's Apology*, Hardy suggests that mathematical beauty arises from an element of surprise. "The most common instance of beauty in mathematics is a brilliant step in an otherwise undistinguished proof." writes Rota (1997), who otherwise disagrees with Hardy: "True, the beauty of a piece of mathematics is often perceived with a feeling of pleasant surprise, which is a way of acknowledging the unexpectedness of an argument; nonetheless, one can find instances of very surprising results which no one has ever thought of classifying as beautiful." The surprise may however be linked to insight. To this end Rota argues that the notion of "mathematical beauty" is a cover up by mathematicians:

"Mathematicians may say that a theorem is beautiful when they really mean to say that the theorem is enlightening." The reason why mathematicians avoid the notion of "enlightenment" is that it admits degrees; some statements are more enlightening than other. "Mathematicians universally dislike any concepts admitting degrees, and will go to any length to deny the logical standing of any such concepts." Beauty on the other hand, like mathematical truth neither admit degrees. Mathematics' reputation for clarity and rigor is largely based on the public's perception that logical deduction is the prototype for rational reasoning; that the main driving force for mathematics is truth. "Mathematical truth seems to be endowed with an absoluteness that few other phenomena of the world can hope to match. On closer inspection, however, one realizes that this definitiveness needs to be tempered down." Rota (1997) argues that "Enlightenment, not truth, is what the mathematician seeks", logical verification does not explain how one statement relates to another, how relevant a statement is – "the mere logical truth of a statement does not enlighten us to the sense of the statement". "The property of being enlightening is objectively attributed to certain mathematical statements and denied to others. [..] Enlightenment is a quality of mathematical statements that one sometimes understands and one sometimes misses, like truth. A mathematical theorem may be enlightening or not, just as it may be true or false."

The Density of Mathematical Proofs

In trying to map Goodman's theory of symbols onto mathematical proofs, it is not clear how to define syntactic types of the proof and to define the boundaries of the symbolic or notational system of the proof – are we just talking about the symbol "=", the two equations (i) and (ii), or is it just the proof (as it is printed above on paper) or is it about the formal (mathematical) system that underlie the proof? What does Goodman's syntactic/semantic density refer to in the context of mathematical proofs?

In mathematical logic, a (formal) theory is a set of sentences expressed in a formal language. *Axioms* of a theory are statements that are included without proof and *theorems* are statements implied by the axioms. A set of axioms is *complete* if, for any statement developed on the basis of the axioms, either that statement or its negation is provable from the axioms. A set of axioms is said to be *consistent* if there is no statement so that both the statement and its negation are provable from the axioms. A formal theory is said to be *effectively generated* if its set of axioms is a recursively enumerable set. This means that there is a computer program that, in principle, could enumerate all the axioms of the theory without listing any statements that are not axioms. This is equivalent to the ability to enumerate all the theorems of the theory without enumerating any statements that are not theorems. The Austrian logician Kurt Gödel (1906-1978) proved with his *first incompleteness theorem* that any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there are arithmetical statements that are true, but not provable in the theory. In other words, although we could prove consistency, one could not formalise it within the language of arithmetic. If there are results that are beyond the reach of a particular formal system, adding further axioms to it would seem to provide a way forward. This however would only lead to an infinite regress in that the new system may have results that are true but cannot be proven with it. Considering the list of all mathematical statements within a formal theory (in the sense of Kurt Gödel's incompleteness theorem), there will statements whose truth – and hence semantic type – we cannot discern and therefore the formal theory fails to be "semantically finitely differentiated" in the sense of Goodman' aesthetics.

In his proof of the incompleteness theorem, Kurt Gödel introduced a function that assigns to each symbol and well-formed formula of some formal language a unique natural number, called its Gödel number. Gödel specifically used this scheme at two levels: first, to encode sequences of symbols representing formulas, and second, to encode sequences of formulas representing proofs. This allowed him to show a correspondence between statements about natural numbers and statements about the provability of theorems about natural numbers, the key observation of the proof. The idea makes use of the fundamental theorem of arithmetic, which says that every positive integer has a unique prime factorization. This syntactic uniqueness of formulas corresponds to Goodman's idea of characters being "syntactically finitely differentiable" (opposed to being "dense").

According to Goodman *notation* is a *symbol system* where each symbol corresponds to one item in the field of reference, and each item corresponds to only one symbol. For a symbol systems to be notational: the characters must be: 1) unambiguous; 2) the characters must be semantically disjoint (meanings cannot intersect); and 3) the system must be finitely differentiated. A musical score (in its entirety) is for Goodman a character in a notational system if and only if "it determines which performances belong to the work, and at the same time, is determined by each of those performances" (Goodman 1976, page 129-30). There are two syntactical rules to which a scheme must adhere in order to be notational: The first rule is that all members of a character are interchangeable, i.e., there's "character indifference," and they're disjoint. He gives a musical score as an example: any quarter-note symbol can be exchanged with any other (Goodman 1976, page 132-34). The second syntactical rule is that characters' disjointness should be testable. That means that characters should be "finitely differentiable." (It is always possible to know to which item a symbol refers). This rule then excludes "dense" systems (like painting) where any two characters can have infinitely more characters between them.

Now, I admitted before that I struggled to go beyond the distinction of denotation and exemplification in applying Goodman's theory of symbols to mathematics. The present section should therefore only be seen as an indication in which directions a further discussion could go. A more detailed discussion would require a deeper understanding of Goodman's aesthetics and is left for future work.

Conclusions

I was not able to map all of Goodman's symptoms of the aesthetic onto mathematics or the notion of proofs in a satisfactory way. However, in my view the distinction between denotation and exemplification in the proof for the irrationality of $\sqrt{2}$ supports Nathalie Sinclair's (and thus Poincaré's) arguments about the *generative* role of the aesthetics. The proof also demonstrates an evaluative role of the aesthetic. Finally, I also suggested that the notion of exemplification in the sense of Goodman reveals *ambiguity* and thus provides support for the arguments of William Byers (2007).

The essence of my discussion is that there is an aesthetic element in mathematical reasoning: *Mathematical reasoning is the art of making appropriate choices*. To this we may add a quote attributed to Pablo Picasso: *Art is a lie that makes us realise truth*. No less artistically but more formally we may conjecture the formula

$$\text{Maths} = \text{truth} + \text{beauty}.$$

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[i] Quoted in J.D. Barrow, *Impossibility*, Oxford University Press, 1998.
[ii] Cited in H.E. Huntley, *The Divine Proportion*, Dover, 1970.
[iii] Readers interested in poetry might enjoy the collection of maths-related poems by Glaz and Growney (2008).
[iv] Proof by contradiction means the proposition is proved by assuming that the opposite of the proposition is true and showing that this assumption is false, which means that the proposition must be true.
[v] The definition should be read as a *process*: "squaring a whole number and multiplying it by two, gives an even number".
[vi] Proving „If *P*, Then *Q*", the contrapositive method assumes Not-*Q* and prove Not-*P*. Compare this to the method of contradiction were we assume *P* and Not-*Q* and prove some sort of contradiction.
[vii] What may seem obvious now, wasn't at the time when irrational numbers we not known – 550 BC the square root of two was not a irrational number, it was simply irrational and "beyond belief".
[viii] For those who don't find everything obvious in this proof, and worry that this is a weakness, may be consoled by reading a more thorough presentation of the proof by Gowers (2002).
[ix] Quoted in Sinclair (2004).
[x] Rota (1997) gives examples.
[xi] This essay is inspired by a talk of Gerhard Heinzmann on 14th October in the Universitätsbuchhandlung Weiland, Kröpeliner Str. in Rostock. In his talk he used a proof for the irrationality of the square root of two to demonstrate aesthetic elements in the sense of the philosopher Nelson Goodman.
[xii] The presentation of Gerhard Heinzmann stopped here but sparked my interest, which led to this essay.
[xiii] "According to Carnap (1971, §§2-3), explicating a term means replacing a pretheoretical term ("explicandum") by another term ("explicatum") that is more exact and embedded in theory." (Vermeulen *et al.* 2009).
[xiv] A string of words is said to be ambiguous if it can be understood as a meaningful sentence in two or more different ways. In our context, ambiguity implies the existence of multiple, alternative frames of reference.
[xv] One might further argue that contradiction played a useful role in the proof.
[xvi] On the surface it appears that there is a difference to Byers, in that our discussion would define ambiguity as implying *alternative* frames of reference, while for Byers (2007, page 28) ambiguity implies mutually *incompatible* or *conflicting* interpretations.
[xvii] Apparently Henri Poincaré was among the first to articulate a conventionalist view. He held the view that axioms in geometry should be chosen for the results they produce, not for their apparent coherence with human intuitions about the physical world.
[xviii] While logic has an essential role in mathematics, it is not the defining property of mathematics. See Byers (2007, page 258) for a discussion of the relationship of logic and mathematics. According to Byers, "logic organizes, stabilizes and communicates ideas" but "logical arguments do not generate ideas."
[xix] Consider for example the Pythagoras theorem (The square of the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides) for which hundreds of independent proofs have been published. In Dreyfus (1986) five alternative proofs for the irrationality of the square root of two are given.